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Report Number 78-20
November 14, 1978

(NASA-CR-185755) CYCLIC ODD-EVEN REDUCTION FOR SYMMETRIC CIRCULANT MATRICES (ICASE) 27 p N89-71350

Unclas
00/64 0224320

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NASA Langley Research Center, Hampton, Virginia

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CYCLIC ODD-EVEN REDUCTION FOR SYMMETRIC CIRCULANT MATRICES

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ABSTRACT

A cyclic odd-even reduction algorithm for symmetric circulant matrices is defined and a subclass of problems to which the algorithm may be applied is identified. Also, sufficient conditions are established which guarantee that the off-diagonal elements of the reduced matrices converge quadratically to zero.

* This report was prepared as a result of work performed under NASA Contract No. NAS1-14101 while the author was in residence at ICASE, NASA Langley Research Center, Hampton, VA 23665, and also under the auspices of the U.S. Department of Energy under Contract No. W-7405-Eng-48.

I. Introduction

Recently there has been a considerable effort made to formulate, analyze and compare methods for solving linear systems of equations on vector or parallel processing computers [2]. Cyclic odd-even reduction has been found to be an effective method for solving certain types of tridiagonal linear systems on vector processors [4, 5].

In a recent paper, Rodrigue, Madsen and Karush [7] developed a generalization of the odd-even reduction algorithm which is applicable to general banded systems of linear equations. Sufficient conditions were established which guaranteed that a single odd-even reduction step could be performed. However, no class of non-tridiagonal matrix problems has been identified and no conditions have been established which guarantee that the odd-even reduction algorithm can be applied in a cyclic fashion in order to fully solve the original banded linear equation problem. This is essential if the algorithm is to be of any practical use.

In this paper we will demonstrate how to modify the Rodrigue, Madsen and Karush algorithm so that it can be applied to symmetric circulant matrix problems. Moreover, we will establish that this algorithm can be applied in a cyclic manner to a subclass of nonsingular symmetric circulant matrix. Also, we will establish conditions which are sufficient to guarantee that the off-diagonal elements of the reduced matrices converge quadratically to zero. This property has been previously established for certain classes of tridiagonal matrix problems [3] and can be used to terminate the solution process early, thereby saving some computation time.

We will adopt the same vector and matrix notation as in [6 , 7]. Of primary use will be that an $n \times n$ real matrix A will be denoted by

$$A = (\underline{a}_j) \quad \text{for} \quad -(n-1) \leq j \leq n-1$$

where for $j > 0$, the vector $\underline{a}_j$ is the j-th superdiagonal of A and the vector $\underline{a}_{-j}$ is the j-th subdiagonal of A. Equality, multiplication, addition and division of equal length vectors are defined componentwise in the obvious manner. When a vector is set equal to a scalar e.g. $\underline{a}_j = c$ we mean that each component of $\underline{a}_j$ equals c.

II. Class of Problems

One of the purposes of this paper is to identify a class of linear equation problems for which cyclic odd-even reduction may be used as a solution process. This class of problems is a subset of the collection of real square matrices which are commonly known as circulant matrices. Matrices of the type

$$A = \begin{bmatrix} c_0 & c_1 & & \cdots & c_{n-1} \\ c_{n-1} & c_0 & c_1 & & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & & c_{n-3} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ c_1 & c_2 & \cdots & & c_0 \end{bmatrix}$$

where the c_i are real numbers are defined to be real circulant matrices. Circulants occur in a variety of applications in solving physical problems [1]. Clearly any $n \times n$ circulant matrix has constant diagonals and is determined by the n-tuple of numbers $(c_0, c_1, \ldots, c_{n-1})$. Using the notation of the previous section, a matrix $A = (\underline{a}_j)$, $n - 1 \leq j \leq n - 1$ is a circulant matrix if

$$\underline{a}_{-j} = c_{n-j} \qquad j = 1, \ldots, n-1$$

and

$$\underline{a}_j = c_j \qquad j = 0, 1, \ldots, n-1 \ ,$$

where $(c_0, c_1, \ldots c_{n-1})$ is any n-tuple of real numbers.

For circulant matrices it is quite convenient to modify slightly the matrix notation of the previous section so that each of the vectors used to define the matrix has the same length. So if $A = (\underline{a}_j), -(n-1) \leq j \leq n-1$, is a circulant matrix we extend the definition of the vectors $\underline{a}_j$ as shown below so that each vector will now have length n

$$\begin{array}{ccccccc}
\underline{a}_0 & \underline{a}_1 & \underline{a}_2 & & \underline{a}_{n-3} & \underline{a}_{n-2} & \underline{a}_{n-1} \\
\searrow & \searrow & \searrow & & \searrow & \searrow & \searrow
\end{array}$$

$$\begin{bmatrix}
c_0 & c_1 & c_2 & \cdots & c_{n-3} & c_{n-2} & c_{n-1} \\
c_{n-1} & c_0 & c_1 & & & c_{n-3} & c_{n-2} \\
c_{n-2} & c_{n-1} & c_0 & & & & c_{n-3} \\
\vdots & & & & & & \vdots \\
 & & & & & & c_2 \\
c_2 & & & & & c_0 & c_1 \\
\cdot & & & & & & \\
c_1 & c_2 & \cdots & c_{n-3} & c_{n-2} & c_{n-1} & c_0
\end{bmatrix}$$

With this extension it is clear that an $n \times n$ matrix $A = (\underline{a}_j)$ is a circulant matrix determined by the n-tuple $(c_0, c_1 \dots c_{n-1})$ if and only if $\underline{a}_j = c_j$ for $j = 0, 1, 2, \dots, n-1$. We note that each diagonal vector except $\underline{a}_0$ can now be identified by two names, i.e. $\underline{a}_j$ or $\underline{a}_{-(n-j)}$, and we will use whichever is the more convenient. To simplify the notation, since each vector $\underline{a}_j$ has all components equal, we will drop the distinction between the scalar component and the vector itself and simply denote either or both by a_j.

It is obvious that the sum of two circulant matrices is also circulant. It is not quite as evident that the product AB of two circulant matrices A and B is also circulant. However, this can be easily seen by writing $A = (a_j)$, $-(n-1) \leq j \leq n-1$ as the sum of n circulant matrices A_k, $k = 0, 1, \dots, n-1$ where A_k has its only nonzero entries given by the diagonal a_k. Each A_k is simply a scalar multiple of a cyclic permutation of the rows of the $n \times n$ identity matrix. We now write

$$AB = (A_0 + A_1 + \cdots + A_{n-1}) \cdot B = A_0 B + A_1 B + \cdots + A_{n-1} B \ .$$

Each of the terms $A_k B$ is a circulant because multiplying by A_k simply scales the entire matrix B and cyclically permutes its rows. Thus AB must be a circulant matrix and we have the following lemma.

<u>Lemma 1</u>:

If A and B are circulant matrices and $C = AB$, then C is a circulant matrix.

It will be useful to develop a simple formula using the current notation for computing the product of two circulant matrices. If $A = (a_j)$, $B = (b_j)$, and $C = (c_j)$ for $-(n-1) \leq j \leq (n-1)$ are circulant matrices such that $C = AB$, then C may be computed by the following formula:

$$c_j = \sum_{k=0}^{n-1} a_k b_{j-k} \quad \text{for } j = 0, 1, \ldots, n-1 . \tag{1}$$

Since C is completely determined from the n scalar entries of its first row, this formula is most easily verified by just computing these values. This result also may be verified by applying the matrix multiplication by diagonals algorithm of Madsen, Rodrigue and Karush [6] to circulant matrices. We note that the diagonals of A and B which contribute to forming the j-th diagonal c_j of C have respective subscripts whose sum is j.

Shifted-Symmetric Circulant Matrices

We will primarily be dealing with the subclass of _symmetric_ circulant matrices ($a_j = a_{-j}$ for $j = 0, 1, \ldots, n-1$). However, there are two other subclasses of circulant matrices which will be very useful. Let $A = (a_j)$ be an $n \times n$ real circulant matrix. We define A to be _upper_ or _lower_ _shifted-symmetric_ if and only if

$$a_j = a_{-j+1} \quad \text{or} \quad a_{-j} = a_{j-1} \text{ respectively,}$$

$$\text{for } j = 1, 2, \ldots, n-1 .$$

Shifted-symmetric circulant matrices have the following properties.

Lemma 2:

Let A and B be upper (lower) shifted-symmetric circulant matrices, then $C = A + B$ is also an upper (lower) shifted-symmetric matrix.

Proof:

Obvious.

Lemma 3:

Let A be an upper (lower) shifted-symmetric circulant matrix and let B be a symmetric circulant matrix. If $C = AB$ or $C = BA$, then C is an upper (lower) shifted symmetric circulant matrix.

Proof:

We will prove only the case where $C = AB$ and A is upper shifted-symmetric as the other results follow in an almost identical way. We first note that from lemma 1 C will be a circulant matrix. Next we suppose that A is such that its only nonzeros lie on the diagonals a_p and a_{-p+1} where $a_p = a_{-p+1}$ from shifted-symmetry. From (1) and the fact that $a_{-p+1} = a_{n-(p-1)}$ we have

$$c_j = \sum_{k=0}^{n-1} a_k b_{j-k} = a_p b_{j-p} + a_{n-(p-1)} b_{j-n+p-1}$$

and since C is a circulant

$$c_{-j+1} = c_{n-(j-1)} = \sum_{k=0}^{n-1} a_k \cdot b_{n-(j-1)-k} = a_p b_{n-j+1-p} + a_{n-(p-1)} b_{n-(j-1)-n+(p-1)}$$

$$= a_p b_{-(j-n-p+1)} + a_{n-(p-1)} b_{-(j-p)}$$

for $j = 0, 1, \dots, n-1$. The symmetry of B now allows us to conclude that $c_j = c_{-j+1}$ or that C is upper shifted-symmetric. In the case (which can occur when n is odd) that a_p and a_{-p+1} are the same diagonal of A we have $n-(p-1) = p$ or $n = 2p-1$ and that

$$c_j = a_p b_{j-p}$$

and

$$c_{-j+1} = a_p b_{n-(j-1)-p} = a_p b_{2p-1-(j-1)-p} = a_p b_{-(j-p)}$$

and again the desired result follows from the symmetry of B. The full proof of lemma 3 now follows easily from lemma 2 and the fact that A is easily decomposed into a finite sum of upper shifted-symmetric matrices of the form above for which we have established the desired results.

Lemma 4:

Let A be a lower shifted-symmetric circulant and let B be an upper shifted-symmetric circulant. If $C = AB$ or $C = BA$, then C is a symmetric circulant.

Proof:

The proof is essentially identical to the proof of lemma 3 and will be omitted.

Lemma 5:

If A is an $n \times n$ lower or upper shifted-symmetric circulant matrix where n is even, then A is singular.

Proof:

If x an n-vector such that $x^T = (1, -1, 1, -1, \dots, -1)$ then a straightforward calculation shows that $Ax = 0$.

III. Basic Algorithm

We will now consider the original problem of solving a linear system of equations $Ax = b$ where A is an $n \times n$ real nonsingular symmetric circulant matrix. We will also assume that $n = 2^q$ for some positive integer $q > 1$. The solution of such matrix problems is often required when numerically solving certain types of elliptic and parabolic partial differential equations when periodic boundary conditions are applied.

The basic idea of cyclic odd-even reduction is to reduce the problem of solving $Ax=b$ to that of solving a smaller problem $(n/2 \times n/2)$ $\bar{A}\bar{x} = \bar{b}$ where $\bar{A}$ has the same basic properties as does A and $\bar{x}$ is subvector consisting of only the even subscripted unknowns of the original vector x. The process is then repeated on the reduced problem $\bar{A}\bar{x} = \bar{b}$. We terminate the process after $q-1$ reduction stages when the remaining system is 2×2.

More specifically we will define a permutation matrix P such that PAP^T has the form

$$PAP^T = \left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right] \begin{array}{l} \} \, n/2 \\ \\ \end{array}$$

where A_1 and A_4 are symmetric circulants and A_2 and A_3 are respectively lower and upper shifted-symmetric circulants. Next, we will constructively show how to define a matrix Q such that

$$QPAP^T = \left[\begin{array}{c|c} \bar{A}_1 & \bar{A}_2 \\ \hline 0 & \bar{A}_4 \end{array}\right]$$

where $\bar{A}_1$ is diagonal, $\bar{A}_2$ is a lower shifted-symmetric circulant and $\bar{A}_4$ is the $n/2 \times n/2$ reduced symmetric circulant matrix. The solution of $Ax=b$ is then equivalent to the solution of the system

$$[QPAP^T]\,Px = QPb$$

which can now be solved by solving the reduced problem $\bar{A}\bar{x} = \bar{b}$.

The matrix P is the $n \times n$ odd-even permutation matrix [7] which is obtained by permuting the rows of the $n \times n$ identity matrix so that the odd numbered rows appear sequentially first and the even numbered rows appear sequentially last. When A is a symmetric circulant matrix of the type we are considering, it not difficult to verify that

$$(2) \qquad PAP^T = \left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right]$$

has the following properties:

1. A_1 and A_4 are $n/2 \times n/2$ symmetric circulants and in fact $A_1 = A_4$,
2. A_2 is an $n/2 \times n/2$ lower shifted-symmetric circulant,
3. A_3 is an $n/2 \times n/2$ upper shifted-symmetric circulant,
4. $A_2^T = A_3$.

Diagonal Elimination

We will now describe how to construct the reduction matrix Q . The matrix Q will be the product of a sequence a matrices Q_i each having the form

$$(3) \qquad Q_i = \left[\begin{array}{c|c} I & T \\ \hline 0 & I \end{array}\right] \begin{array}{l} \left.\vphantom{I}\right\} n/2 \\ \vphantom{I} \end{array}$$

or

$$(4) \qquad Q_1 = \left[\begin{array}{c|c} I & 0 \\ \hline S & I \end{array}\right] \Big\} n/2$$

where T and S are respectively lower and upper shifted symmetric circulants, each having at most two nonzero diagonals.

We will identify the diagonals of the matrices $A_1 - A_4$ of (2) and T and S of (3) and (4) by

$$A_1 = (a_j)\ , -(m-1) \le j \le m-1$$

$$A_2 = (b_j)\ , -(m-1) \le j \le m-1$$

$$A_3 = (c_j)\ , -(m-1) \le j \le m-1$$

$$A_4 = (d_j)\ , -(m-1) \le j \le m-1$$

$$T = (t_j)\ , -(m-1) \le j \le m-1$$

$$S = (s_j)\ , -(m-1) \le j \le m-1 \quad \text{where } m = n/2\ .$$

The reduction process will be constructed in $n/2$ stages, each stage having two steps. For the initial presentation we will assume that PAP^T has no zero entries and that no zeros other than those specifically desired are introduced into the matrix at any stage.

<u>Stage 1</u>

<u>Step 1. Eliminate the diagonal $\bar{a}_{m/2}$ in $A_1^{(1)}$</u>.

We define the matrix Q_1 by letting $t_0 = t_{-1} = -\dfrac{a_{m/2}}{2c_{m/2}}$, and $t_j = 0$ for $j = 1, 2, \ldots, m-2$. Clearly T is a lower shifted-symmetric circulant with two nonzero diagonals. Now we compute

$$Q_1 PAP^T = \left[\begin{array}{c|c} A_1^{(1)} & A_2^{(1)} \\ \hline A_3 & A_4 \end{array}\right]$$

and note that $A_1^{(1)} = A_1 + TA_3$ and $A_2^{(1)} = A_2 + TA_4$. Since T is lower shifted-symmetric and A_4 is symmetric, we have from lemma 3 that $A_2^{(1)}$ is a lower shifted-symmetric circulant as was A_2. Also, from lemma 4 we have that $A_1^{(1)}$ is a symmetric circulant. Moreover, we have from (1) that

$$\begin{aligned} \bar{a}_{m/2} &= a_{m/2} + t_0 c_{m/2} + t_{m-1} c_{m/2-(m-1)} \\ &= a_{m/2} + t_0 c_{m/2} + t_{-1} c_{-m/2+1} \\ &= a_{m/2} + t_0 c_{m/2} + t_{-1} c_{m/2} \\ &= a_{m/2} - \tfrac{1}{2} a_{m/2} - \tfrac{1}{2} a_{m/2} = 0 \end{aligned}$$

so that the diagonal $\bar{a}_{m/2}$ has been eliminated from $A_1^{(1)}$.

Step 2. Eliminate the diagonals $\bar{c}_{m/2}$ and $\bar{c}_{-m/2+1}$ in $A_3^{(1)}$.

(Note that $c_{m/2} = c_{-m/2+1}$.)

We define Q_2 by letting $s_0 = s_1 = -\dfrac{c_{m/2}}{\bar{a}_{m/2-1}}$ (the other diagonals of S being 0) and compute

$$Q_2 Q_1 PAP^T = \left[\begin{array}{c|c} A_1^{(1)} & A_2^{(1)} \\ \hline A_3^{(1)} & A_4^{(1)} \end{array}\right]$$

where $A_3^{(1)} = A_3 + S A_1^{(1)}$ and $A_4^{(1)} = A_4 + S A_2^{(1)}$. As in step 1, $A_3^{(1)}$ and $A_4^{(1)}$ have the same respective symmetry properties as do A_3 and A_4. Also, we have

$$\bar{c}_{m/2} = c_{m/2} + s_0 \bar{a}_{m/2} + s_1 \bar{a}_{m/2-1}$$

$$= c_{m/2} + s_1 \bar{a}_{m/2-1}$$

$$= c_{m/2} - c_{m/2} = 0 \ .$$

From upper shifted-symmetry we have that $\bar{c}_{-m/2+1} = 0$, so we see that Q_2 eliminates two diagonals in $A_3^{(1)}$. This completes Stage 1.

Stage 2

Step 1. Eliminate the diagonals $\bar{a}_{\pm(m/2-1)}$ in $A_1^{(2)}$.

We define the matrix Q_3 by setting $t_0 = t_{-1} = -\dfrac{a_{m/2-1}}{c_{m/2-1}}$

and compute

$$Q_3 Q_2 Q_1 PAP^T = \left[\begin{array}{c|c} A_1^{(2)} & A_2^{(2)} \\ \hline A_3^{(1)} & A_4^{(1)} \end{array}\right]$$

where $A_1^{(2)} = A_1^{(1)} + T A_3^{(1)}$ and $A_2^{(2)} = A_2^{(1)} + T A_4^{(1)}$. Again from lemmas 3 and 4, $A_1^{(2)}$ and $A_2^{(2)}$ have the desired symmetry properties. Computing $\bar{a}_{m/2-1}$ in $A_1^{(2)}$ we have

$$\bar{a}_{m/2-1} = a_{m/2-1} + t_0 c_{m/2-1} + t_{m-1} c_{m/2-1-(m-1)}$$

$$= a_{m/2-1} + t_0 c_{m/2-1} + t_{m-1} c_{-m/2}$$

$$= a_{m/2-1} - a_{m/2-1} + 0 = 0 \ .$$

By symmetry this shows that $\bar{a}_{-(m/2-1)} = 0$. Moreover, the zeros introduced in Stage 1 remain because $A_3^{(1)}$ is upper shifted-symmetric and

$$\bar{a}_{m/2} = a_{m/2} + t_0 c_{m/2} + t_{m-1} c_{m/2-(m-1)}$$

$$= 0 + 0 + t_{m-1} c_{-m/2+1} = 0 .$$

<u>Step 2. Eliminate the diagonals $\bar{c}_{m/2-1}$ and $\bar{c}_{-m/2+2}$ in $A_3^{(2)}$</u> .

We define Q_4 by setting $s_0 = s_1 = -\dfrac{c_{m/2-1}}{\bar{a}_{m/2-2}}$ and compute

$$Q_4 Q_3 Q_2 Q_1 PAP^T = \left[\begin{array}{c|c} A_1^{(2)} & A_2^{(2)} \\ \hline A_3^{(2)} & A_4^{(2)} \end{array}\right]$$

where $A_3^{(2)} = A_3^{(1)} + S A_1^{(2)}$ and $A_4^{(2)} = A_4^{(1)} + S A_2^{(2)}$. As before, $A_3^{(2)}$ and $A_4^{(2)}$ have the desired symmetry properties. We also have

$$\bar{c}_{-m/2+2} = \bar{c}_{m/2-1} = c_{m/2-1} + s_0 \bar{a}_{m/2-1} + s_1 \bar{a}_{m/2-2}$$

$$= c_{m/2-1} + 0 - c_{m/2-1} = 0$$

and

$$\bar{c}_{-m/2+1} = \bar{c}_{m/2} = c_{m/2} + s_0 \bar{a}_{m/2} + s_1 \bar{a}_{m/2-1}$$

$$= 0 + 0 + 0 = 0$$

so the desired diagonals have been eliminated and those previously eliminated remain zero.

Notice that we are establishing a band of zeros ($a_j = 0$ for $m/2 - 1 \le j \le m/2 + 1$ and $c_j = 0$ for $m/2 - 1 \le j \le m/2 + 2$) both above and below the main diagonal.

We now proceed inductively and assume that stage k of the process has been successfully completed and that $A_1^{(k)} = (a_j)$ is a symmetric circulant and that $a_j = 0$ for $m/2 - (k-1) \le j \le m/2 + (k-1)$ and $A_3^{(k)} = (c_j)$ is an upper shifted-symmetric circulant with $c_j = 0$ for $m/2 - (k-1) \le j \le m/2 + k$. We now proceed to the next stage.

Stage $k+1$

Step 1. Eliminate the diagonals $\bar{a}_{\pm(m/2-k)}$ in $A_1^{(k+1)}$.

We define $Q_{2(k+1)-1} = Q_{2k+1}$ by setting $t_0 = t_{-1} = -\dfrac{a_{m/2-k}}{c_{m/2-k}}$

and compute

$$Q_{2k+1} Q_{2k} \cdots Q_1 PAP^T = \left[\begin{array}{c|c} A_1^{(k+1)} & A_2^{(k+1)} \\ \hline A_3^{(k)} & A_4^{(k)} \end{array}\right]$$

where $A_1^{(k+1)} = A_1^{(k)} + T A_3^{(k)}$ and $A_2^{(k+1)} = A_2^{(k)} + T A_4^{(k)}$. From lemmas 3 and 4, $A_1^{(k+1)}$ and $A_2^{(k+1)}$ have the desired symmetry properties. Also

$$\begin{aligned} \bar{a}_{\pm(m/2-k)} &= a_{m/2-k} + t_0 c_{m/2-k} + t_{m-1} c_{m/2-k-(m-1)} \\ &= a_{m/2-k} - a_{m/2-k} + t_{m-1} c_{-m/2-k+1} \\ &= 0 + t_{m-1} c_{m-(m/2+k-1)} = t_{m-1} c_{m/2-k+1} = 0 \end{aligned}$$

from the induction hypotheses. To complete this step we need to verify that $\bar{a}_j = 0$ for $m/2 - (k-1) \leq j \leq m/2 + (k-1)$. Let $\bar{a}_p$ be one of these diagonals. We have

$$\bar{a}_p = a_p + t_0 c_p + t_{m-1} c_{p-(m-1)}$$

$$= 0 + 0 + t_{m-1} c_{m-(m-p-1)} = t_{m-1} c_{p+1} = 0$$

from the induction hypotheses since $m/2 - (k-1) \leq p+1 \leq m/2+k$.

<u>Step 2. Eliminate the diagonals $\bar{c}_{m/2-k}$ and $\bar{c}_{-m/2+k+1}$ in $A_3^{(k+1)}$.</u>

We define Q_{2k+2} by setting $s_0 = s_1 = -\dfrac{c_{m/2-k}}{\bar{a}_{m/2-k-1}}$ and compute

$$Q_{2k+2} Q_{2k+1} \cdots Q_1 PAP^T = \left[\begin{array}{c|c} A_1^{(k+1)} & A_2^{(k+1)} \\ \hline A_3^{(k+1)} & A_4^{(k+1)} \end{array}\right]$$

where $A_3^{(k+1)} = A_3^{(k)} + S A_1^{(k+1)}$ and $A_4^{(k+1)} = A_4^{(k)} + S A_2^{(k+1)}$. Lemmas 3 and 4 allow us to conclude that $A_3^{(k+1)}$ and $A_4^{(k+1)}$ have the desired symmetry properties. Also

$$\bar{c}_{-m/2+k+1} = \bar{c}_{m/2-k} = c_{m/2-k} + s_0 \bar{a}_{m/2-k} + s_1 \bar{a}_{m/2-k-1} = 0 \quad \text{as before.}$$

Now compute $\bar{c}_p$ where $m/2 - (k-1) \leq p \leq m/2+k$. We have

$$\bar{c}_p = c_p + s_0 a_p + s_1 a_{p-1}$$

$$= 0 + 0 + 0 = 0$$

from step 1 and the induction hypotheses.

This completes the description of the basic algorithm. After $m/2$ stages the reduction is complete since $A_3^{(m/2)} \equiv 0$, $A_1^{(m/2)}$ is a diagonal matrix and $A_4^{(m/2)}$ is the reduced symmetric circulant.

The basic algorithm as described above can fail at either step 1 or step 2 of any stage if the diagonal used as a divisor in that step is zero. We will now discuss how to overcome this difficulty. Suppose at stage $k+1$ and step 1 the divisor diagonal $c_{m/2-k} = 0$. In this case we search the remaining diagonals c_j, for $1 \leq j < m/2-k$ for the nonzero diagonal with the largest subscript less than $m/2-k$ and use this as our divisor. Let $c_{m/2-k-p}$, where $1 \leq p \leq m/2-k-1$, be the divisor diagonal. Next we define Q_{2k+1} by setting $t_p = t_{-p+1} = -\frac{a_{m/2-k}}{c_{m/2-k-p}}$ and by setting the remaining diagonals to zero. Now we compute

$$\begin{aligned}
\bar{a}_{\pm(m/2-k)} &= a_{m/2-k} + t_p c_{m/2-k-p} + t_{m-(p-1)} c_{m/2-k-m+(p-1)} \\
&= a_{m/2-k} - a_{m/2-k} + t_{m-(p-1)} c_{-(m/2+k-p+1)} \\
&= 0 + t_{m-(p-1)} c_{m-(m/2+k-p+1)} = t_{m-(p-1)} c_{m/2-k+p-1}
\end{aligned}$$

Since $p \neq 0$, the induction hypothesis may be extended so that $A_3^{(k)} = (c_j)$ is such that $c_j = 0$ for $m/2-(k-1)-p \leq j \leq m/2+k+p$. A routine calculation now shows that $m/2-k+p-1$ lies within this range so we conclude that

$$\bar{a}_{\pm(m/2-k)} = 0 .$$

We have left to show that the previously eliminated diagonals are left undisturbed by this modification. Let $\bar{a}_q$ be such that

$m/2-(k-1) \leq q \leq m/2+(k-1)$. Then,

$$\bar{a}_q = a_q + t_p c_{q-p} + t_{m-(p-1)} c_{q-(m-(p-1))} = t_p c_{q-p} + t_{-p+1} c_{q+p-1} .$$

A simple calculation shows that both subscripts $q-p$ and $q+p-1$ lie within the range of the extended induction hypothesis above so $\bar{a}_q = 0$.

If at some stage the search for a nonzero diagonal of $A_3^{(k)}$ yields the conclusion that $A_3^{(k)} \equiv 0$, then the problem splits into two problems involving the symmetric circulant $A_1^{(k)}$ and $A_4^{(k)}$. Both of these matrices must be nonsingular since A was assumed nonsingular, and the above algorithm can then be applied to each.

If the basic algorithm should fail during stage $k+1$ and step 2, i.e. the divisor diagonal $\bar{a}_{m/2-k-1} = 0$, then we search the remaining diagonals $\bar{a}_j$, for $0 \leq j \leq m/2-k-2$ for the nonzero diagonal with the largest subscript less than $m/2-k-1$ and use this as our divisor. Let $\bar{a}_{m/2-k-1-p}$, where $1 \leq p \leq m/2-k-1$ be this divisor. Q_{2k+2} is defined by setting $s_{p+1} = s_{-p} = -\dfrac{c_{m/2-k}}{\bar{a}_{m/2-k-1-p}}$ and by setting the other diagonals to zero. Completely analogous calculations now show this allows step 2 to be completed as before. If the search for a nonzero diagonal divisor yields the conclusion that $A_1^{(k+1)} \equiv 0$, then we would have to conclude that the matrix $Q_{2k+1} \cdots Q_1 PAP^T$ is singular because it is of the form

$$\left[\begin{array}{c|c} 0 & A_2^{(k+1)} \\ \hline A_3^{(k)} & A_4^{(k)} \end{array}\right]$$

and $A_3^{(k)}$ is upper shifted-symmetric and hence singular by lemma 5. This would lead us to the conclusion that A must be singular which is a contradiction. We must therefore conclude that the search for a non-zero divisor diagonal will never fail in step 2 if A is nonsingular to begin with.

This completes the description of one odd-even reduction step for symmetric circulant matrices. After each reduction, another odd-even reduction step may be performed on the resulting reduced matrix and this defines the process of cyclic odd-even reduction. We terminate the process when the reduced matrix is 2×2 and solve this system explicitly. The final solution is then obtained by a back substitution like process.

We have constructively and inductively proved our main theorem.

<u>Theorem</u>

If A is an $n \times n$ nonsingular, symmetric circulant matrix where $n = 2^p$ for some positive integer $p > 1$, then the linear system $Ax = b$ may be solved by cyclic odd-even reduction.

We remark that the original work of Rodrigue, Madsen and Karush [7] was directed toward the solution of banded matrices on vector processors. In this paper we chose the non-banded setting. However, it should be obvious that the above algorithm is applicable to "banded" symmetric circulants, i.e. $n \times n$ symmetric circulants such that

$$a_{n/2 \pm j} = 0 \quad \text{for} \quad j = 0, 1, 2, \cdots, q \quad \text{where} \quad 0 \leq q < n/2 .$$

It can be shown that in this case, if one starts with a banded symmetric circulant, then the reduced matrix will also be similarly "banded" with at most the same number of nonzero diagonals as were in the original matrix.

IV. Quadratic Convergence

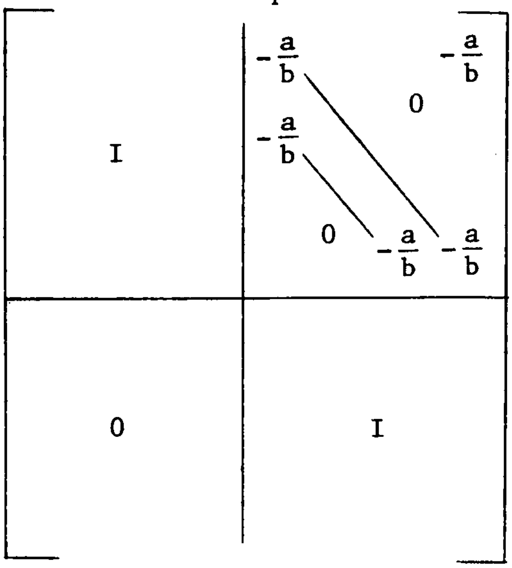
It has been proved [3], under certain dominance conditions, that the off-diagonal elements of the reduced matrices converge quadratically to zero when cyclic odd-even reduction is applied to certain tridiagonal systems. We will now establish sufficient conditions which insure a similar behavior when our cyclic odd-even reduction algorithm is applied to pentadiagonal symmetric circulant matrices.

To illustrate the use of the algorithm, for "banded" matrices we will let A be the $n \times n$ (n even) pentadiagonal symmetric circulant

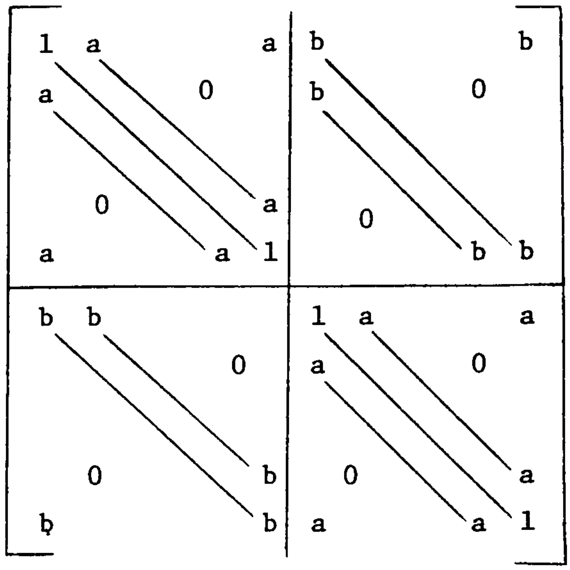
$$A = \begin{bmatrix} 1 & b & a & & & a & b \\ b & \ddots & \ddots & \ddots & & 0 & a \\ a & \ddots & \ddots & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & a \\ a & 0 & & \ddots & \ddots & \ddots & b \\ b & a & & & a & b & 1 \end{bmatrix}$$

The matrix Q_q (the first of the reduction matrices which is not just the identity matrix) and PAP^T are as follows

Q_q

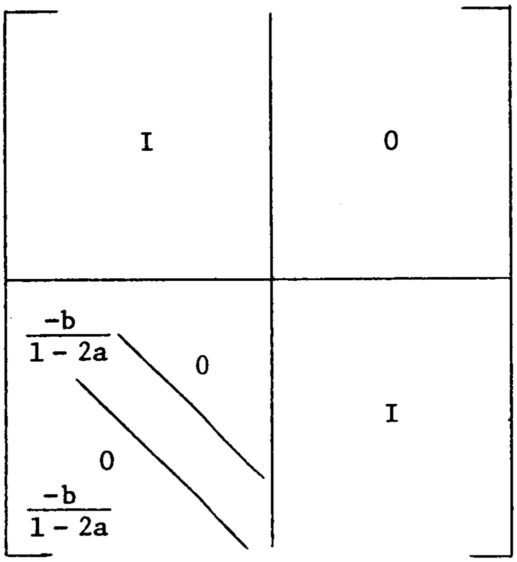


PAP^T

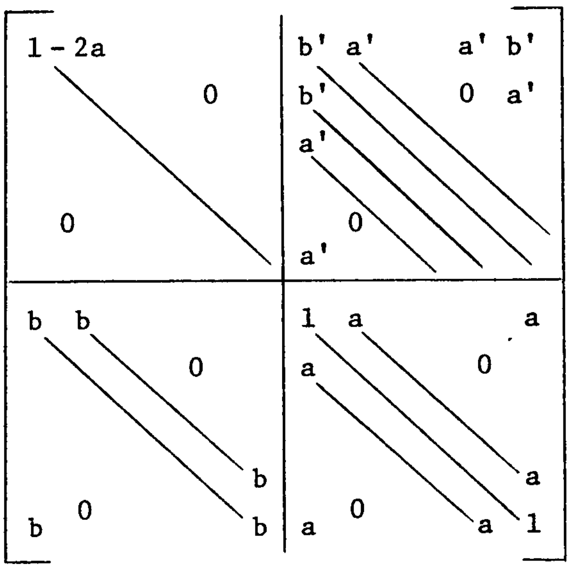


The matrices Q_{q+1} and $Q_q PAP^T$ of the next step are

Q_{q+1}



$Q_q PAP^T$



where $b' = b - \frac{a}{b} - \frac{a^2}{b}$ and $a' = - \frac{a^2}{b}$.

The final matrix $Q_{q+1} Q_q PAP^T$ is shown next. Note that for convenience we have scaled the reduced matrix $\bar{A}_4$ so that its main diagonal is one.

$$
Q_{q+1}\,Q_q P A P^T = \left[\begin{array}{ccccc|cccccc}
1-2a & & & & & b' & a' & & & a' & b' \\
& \ddots & & 0 & & b' & \ddots & \ddots & 0 & & a' \\
& & \ddots & & & a' & \ddots & \ddots & \ddots & & \\
& 0 & & \ddots & & & \ddots & \ddots & \ddots & \ddots & a' \\
& & & & \ddots & a' & & 0 & a' & b' & b' \\
\hline
& & & & & 1 & \bar{b} & \bar{a} & & \bar{a} & \bar{b} \\
& & & & & \bar{b} & \ddots & \ddots & \ddots & 0 & \bar{a} \\
& & 0 & & & \bar{a} & \ddots & \ddots & \ddots & \ddots & \\
& & & & & & \ddots & \ddots & \ddots & \ddots & \bar{a} \\
& & & & & \bar{a} & 0 & \ddots & \ddots & \ddots & \bar{b} \\
& & & & & \bar{b} & \bar{a} & & \bar{a} & \bar{b} & 1
\end{array}\right]
$$

where b' and a' are as before and

$$
\bar{b} = \frac{2a - b^2}{1 - 2b^2 + 2a^2}
$$

(5)

$$
\bar{a} = \frac{a^2}{1 - 2b^2 + 2a^2} \quad .
$$

If we denote the diagonals of each succeeding reduced matrix by a_i and b_i for $i = 1, 2, \ldots,$ with $a_0 = a$ and $b_0 = b$, then from (5) we see that these diagonals satisfy the following recursion relations (until the reduced matrix becomes less than 8×8 in size)

$$
a_{i+1} = \frac{a_i^2}{1 - 2b_i^2 + 2a_i^2}
$$

$$
b_{i+1} = \frac{2a_i - b_i^2}{1 - 2b_i^2 + 2a_i^2} \quad .
$$

We now state and prove a quadratic converge theorem concerning these diagonals.

Theorem

If $0 \leq a_0 < \frac{1}{8}$ and $|b_0| < \frac{1}{2}$, then

$$a_i \leq \tfrac{1}{2}\left(\frac{a_0}{\frac{1}{2}}\right)^{(2^i)} \quad \text{and} \quad |b_i| \leq \tfrac{1}{2}\left(\frac{\max(2a_0, b_0^2)}{(\frac{1}{2})^2}\right)^{(2^{i-1})} .$$

Proof:

The proof will be by induction. For $i = 1$ we have

$$a_1 = \frac{a_0^2}{1 - 2b_0^2 + 2a_0^2} \leq \frac{a_0^2}{1 - 2b_0^2} \leq \frac{a_0^2}{\frac{1}{2}} = \tfrac{1}{2}\left(\frac{a_0}{\frac{1}{2}}\right)^2 ,$$

$$b_1 = \frac{2a_0 - b_0^2}{1 - 2b_0^2 + 2a_0^2} \leq \frac{2a_0}{\frac{1}{2}} = \tfrac{1}{2}\left[\frac{2a_0}{(\frac{1}{2})^2}\right] ,$$

and

$$b_1 = \frac{2a_0 - b_0^2}{1 - 2b_0^2 + 2a_0^2} \geq \frac{-b_0^2}{1 - 2b_0^2 + 2a_0^2} \geq \frac{-b_0^2}{\frac{1}{2}} = -\tfrac{1}{2}\left[\frac{b_0^2}{(\frac{1}{2})^2}\right]$$

or

$$|b_1| \leq \tfrac{1}{2}\left(\frac{\max(2a_0, b_0^2)}{(\frac{1}{2})^2}\right)^{(2^0)} .$$

For convenience we define $d_{i+1} = 1 - 2b_i^2 + 2a_i^2$ and note that $d_1 \geq \frac{1}{2}$. We assume that the results are true for i and will show that this implies they are also true for $i+1$. We include as part of the induction that $d_i \geq \frac{1}{2}$. First, we have

$$d_{i+1} = 1 - 2b_i^2 + 2a_i^2 \geq 1 - 2b_i^2 \geq 1 - 2(\tfrac{1}{2})^2 \left(\frac{\max(2a_0, b_0^2)}{(\frac{1}{2})^2} \right)^{(2^i)}$$

$$\geq 1 - 2(\tfrac{1}{2})^2 = \tfrac{1}{2} .$$

Also,

$$a_{i+1} = \frac{a_i^2}{1 - 2b_i^2 + 2a_i^2} \leq \frac{\left[\frac{1}{2} \left(\frac{a_0}{\frac{1}{2}} \right)^{(2^i)} \right]^2}{\frac{1}{2}} = \tfrac{1}{2} \left(\frac{a_0}{\frac{1}{2}} \right)^{(2^{i+1})} ,$$

$$b_{i+1} = \frac{2a_i - b_i^2}{1 - 2b_i^2 + 2a_i^2} \leq \frac{2a_i}{\frac{1}{2}} \leq 2 \left[\left(\frac{a_0}{\frac{1}{2}} \right)^{(2^i)} \right] = 2(2a_0)^{(2^i)}$$

$$\leq \tfrac{1}{2} \left(\frac{\max(2a_0, b_0^2)}{(\frac{1}{2})^2} \right)^{(2^i)} ,$$

and

$$b_{i+1} \geq \frac{-b_i^2}{\frac{1}{2}} \geq \frac{-\left[\frac{1}{2} \left(\frac{\max(2a_0, b_0^2)}{(\frac{1}{2})^2} \right)^{(2^{i-1})} \right]^2}{\frac{1}{2}}$$

$$= -\tfrac{1}{2} \left(\frac{\max(2a_0, b_0^2)}{(\frac{1}{2})^2} \right)^{(2^i)}$$

or

$$|b_{i+1}| \leq \tfrac{1}{2}\left(\frac{\max(2a_0, b_0^2)}{(\frac{1}{2})^2}\right)^{(2^i)}$$

which completes the induction and shows that the off-diagonal elements converge quadratically to zero under the hypotheses of the theorem.

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